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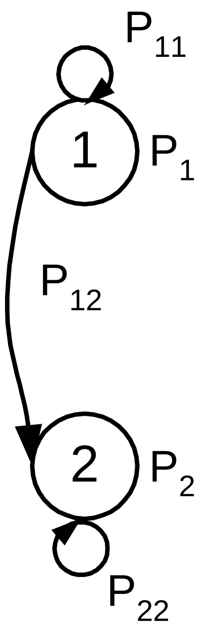
Reliability of Technical Systems

Markov Modeling for Reliability Analysis

Introduction: „non-repairable unit"

System description: A unit changes its state at time *t* from „functional" to „failed“.

This gives us the „state diagram“



state 1: „function"; state 2: „failure"

P_1: P_2: probability, that the unit is in state 1 at time *t* or interval $t+\Delta t$ (and P_2 accordingly)

- P_1: $P_1(t+\Delta t) = [P_1(t) \wedge P_{11}(\Delta t)]$
- P_2: $P_2(t+\Delta t) = [P_1(t) \wedge P_{12}(\Delta t)] \vee [P_2(t) \wedge P_{22}(\Delta t)]$

P_{11}: $P_{11}(\Delta t)$: Probability, that a unit stays in state 1 during the time interval *t* to $t+\Delta t$ (and P_{22} accordingly)

P_{12}: $P_{12}(\Delta t)$: State change probability, that an unit changes its state from 1 to 2 during the time interval *t* to $t+\Delta t$

Setting the system of equations

Assumption:
- Failure rate λ = constant $\Rightarrow$ Exponential distribution; the reciprocal value from λ is named Mean Time to Failure (MTTF)
- Failure probability at time $t = 0$: $F(t = 0) = 0$

$\lambda \cdot \Delta t$: Probability, that the unit fails between time t und $t+\Delta t$.

System of equations

$P_1(t+\Delta t) - P_1(t) = -\lambda \Delta t P_1(t)$

$P_2(t+\Delta t) - P_2(t) = +\lambda \Delta t P_1(t)$

This system of equations as system of differential equations

$$\frac{d}{dt} P_1(t) = -\lambda \cdot P_1(t)$$

$$\frac{d}{dt} P_2(t) = +\lambda \cdot P_1(t)$$

This system in Matrix form:

$$\frac{d}{dt}\begin{pmatrix} P_1(t) \\ P_2(t) \end{pmatrix} = \begin{pmatrix} -\lambda & 0 \\ \lambda & 0 \end{pmatrix} \cdot \begin{pmatrix} P_1(t) \\ P_2(t) \end{pmatrix}$$

Failure density function: $$f(t) = \frac{dF(t)}{dt} = -\frac{dR(t)}{dt}$$

Exponential distribution:
$$F(t) = 1 - \exp(-\lambda t);\ R(t) = \exp(-\lambda t)$$
$$f(t) = \lambda \cdot \exp(-\lambda t)$$

with

$$P_i = \int \frac{dP_i}{dt} \cdot dt$$

hence, with P_1 as the survival probability *R(t)* the system of equations becomes

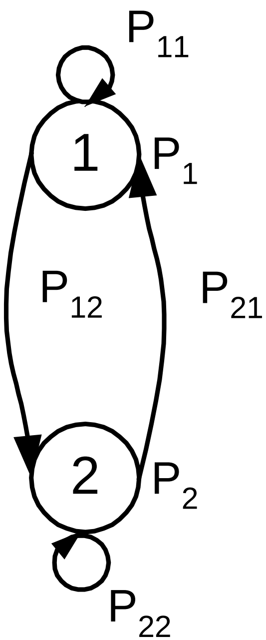
$$\frac{d}{dt}P_1(t) = -\lambda \cdot P_1(t) = -\lambda \cdot \exp(-\lambda t)$$
$$\frac{d}{dt}P_2(t) = +\lambda \cdot P_1(t) = \lambda \cdot \exp(-\lambda t)$$

Through standard integration we arrive at

$$P_1(t) = exp(-\lambda t)$$

$$P_2(t) = 1 - exp(\ \lambda t)$$

State diagram of a repairable unit



State 1: "Function"; State 2: "in Repair"

P_1: P_1: State probability that the unit is at time *t* or time interval $t = t + \Delta t$ in State 1 (P_2: respectively)

- P_1: $P_1(t+\Delta t) = [P_1(t) \wedge P_{11}(\Delta t)] \vee [P_2(t) \wedge P_{21}(\Delta t)]$
- P_2: $P_2(t+\Delta t) = [P_1(t) \wedge P_{12}(\Delta t)] \vee [P_2(t) \wedge P_{22}(\Delta t)]$

P_{11}: $P_{11}(\Delta t)$: Probability, that the unit is in the time interval *t* to $t+\Delta t$ in state 1 (P_{22}: respectively)

P_{12}: $P_{12}(\Delta t)$: State change probability that the unit changes in time interval *t* to $t+\Delta t$ from state1 to state 2, thus fails.

P_{21}: $P_{21}(\Delta t)$: State change probability that the unit changes in time interval *t* to $t+\Delta t$ from state 2 to state 1, thus is repaired.

Extended System of Equations

with μ as the repair rate

$$\frac{d}{dt}P_1(t) = -\lambda \cdot P_1(t) + \mu \cdot P_2(t)$$
$$\frac{d}{dt}P_2(t) = +\lambda \cdot P_1(t) - \mu \cdot P_2(t)$$

This system can be solved through a La Place Transformation. Furthermore, several software packages are available for the design and calculation of state diagrams, e.g. CARMS[1] (see RSN-Website, freeware).

Repair rate $\mu(t)$

The definition of the repair rate μ corresponds to the failure rate λ and is here assumed to be constant (and hence the repair probability becomes $FR(t)$ = 1-exp[-$\mu \cdot t$]). The reciprocal of μ is MTTR (Mean Time to Repair), which can be estimated through the empirical average repair time.

[1] Computer-Aided Rate Modeling and Simulation

Simplification: State Probability for $t \to \infty$

Advantage: $\frac{d}{dt} P_i(t) = 0$

That means that the gradient of the Reliability function $P_i(t)$ is quasi zero after „infinite time".

Example

A repairable unit

Simplified system of equations

$$0 = -\lambda \cdot P_1(\infty) + \mu \cdot P_2(\infty)$$

$$0 = +\lambda \cdot P_1(\infty) - \mu \cdot P_2(\infty)$$

This is not enough in order to solve the system, because the *Nth* state is derived from the previous states.

The missing information is the constraint: $\sum_i P(\infty)_i = 1$

For the system of equations (1), this means :

$$0 = 1 - P_1(\infty) - P_2(\infty)$$
$$0 = \lambda \cdot P_1(\infty) - \mu \cdot P_2(\infty)$$

or

$$0 = -\lambda \cdot P_1(\infty) + \mu \cdot P_2(\infty)$$
$$0 = 1 - P_1(\infty) - P_2(\infty)$$

Solving the left equation for $P_2(\infty) = 1 - P_1(\infty)$

and replacing on the right gives an equations that is solvable for $P_1(\infty)$

$$0 = -\lambda \cdot P_1(\infty) + \mu \cdot \left(1 - P_1(\infty)\right)$$

Solving this equation gives the state probaility for the state 1 "Function" – in other words the stationary availability

$$P_1(\infty) = \frac{\mu}{\lambda + \mu}$$

The stationary failure probability is then

$$P_2(\infty) = 1 - P_1(\infty) = \frac{\lambda}{\lambda + \mu}$$

Data sources

- specific Data

  Available data for a specific unit; its validity hence is provided.

  This kind of data is ideal for a reliability analyis. Nevertheless, often there is a lack of it in practice.

- generic Data

  Such data often are given in publications. The validity of this data is however not provided.

  Application to other units is questionable; convenient increase of the data basis

- „expert judgement"

  subjective judgement of an expert regarding the unit behavior.

  Rather inappropriate for a reliability analysis, but often the only available data source.